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# The $E_8$ geometry from a Clifford perspective

Pierre-Philippe Dechant

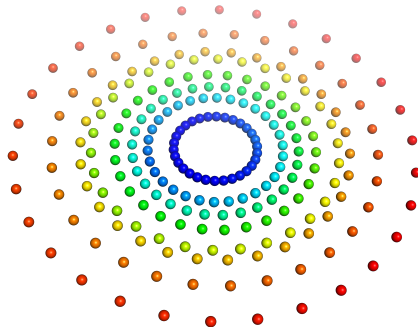
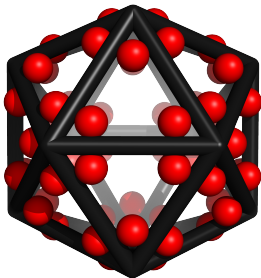
Mathematics Department, University of York

AGACSE Barcelona – July 29, 2015

# AGACSE 2012, La Rochelle: The birth of the $E_8$ question...

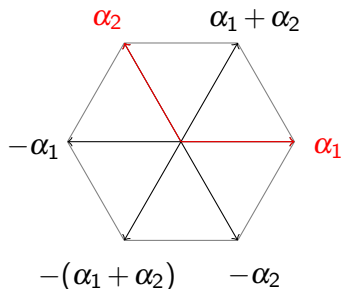


# All exceptional geometries from 3D geometry





## Root systems – $A_2$



reflection/Coxeter groups

**Root system**  $\Phi$ : set of vectors  $\alpha$  such that

1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

**Simple roots**: express every element of  $\Phi$  via a  $\mathbb{Z}$ -linear combination (with coefficients of the same sign).

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

# Cartan Matrices

Cartan matrix of  $\alpha_i$ s is 
$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

**Coxeter-Dynkin diagrams:** node = simple root, no link = roots orthogonal, simple link = roots at  $\frac{\pi}{3}$ , link with label  $m$  = angle  $\frac{\pi}{m}$ .

$$A_3 \circ - \circ - \circ$$

$$B_3 \circ - \overset{4}{\circ} - \circ$$

$$H_3 \circ - \overset{5}{\circ} - \circ$$

$$I_2(n) \circ - \overset{n}{\circ}$$

# Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = **group** and **manifold** (e.g. spin groups: Doran, Hestenes et al)
- Lie algebra = **bilinear, antisymmetric bracket** and **Jacobi identity** (e.g. bivector algebras) = Lie group near the identity
- 'Nice' Lie algebras have **triangular decomposition**:

$$\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+ : SU(2) : 1 + 1 + 1, E_8 : 120 + 8 + 120$$

- $\mathcal{H}$ : **Cartan subalgebra/QN**; Creation & annihilation ops  $\mathcal{N}$ : **root lattice from crystallographic root systems**
- **Weyl** group is a **crystallographic** Coxeter group:

$$A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8.$$

- So via this route always crystallographic! **Neglect**  $I_2(n), H_3, H_4$ .

## Example – $A_1$ , $SU(2)$ , Angular Momentum



- Cartan subalgebra = Quantum number:  $L_z$
- $\mathcal{N}_+$ : raising operator  $L_+ = \alpha$
- $\mathcal{N}_-$ : lowering operator  $L_- = -\alpha$
- ( $L^2$  is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

## Example – $A_1$ , $SU(2)$ , Electroweak

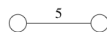
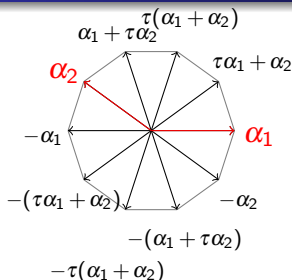
$W^+$



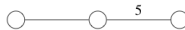
$W^-$

- Cartan subalgebra – Quantum number:  $A$
- $\mathcal{N}_+$ : raising operator  $W^+ = \alpha$
- $\mathcal{N}_-$ : lowering operator  $W^- = -\alpha$
- (Since SM electroweak is actually  $SU(2) \times U(1)$ ,  $U(1)$  gives another field  $i$ , such that physical  $Z^0$  and  $\gamma$  are superpositions of  $A$  and  $i$ )
- Also  $W^\pm$  now charged and self-interact, unlike QED

# Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$ : 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5**

linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio}$$

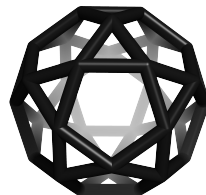
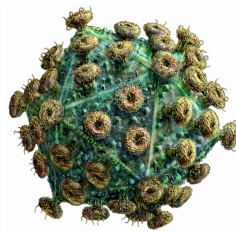
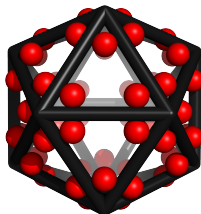
$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

$$\tau + \sigma = 1, \tau\sigma = -1$$

# The Icosahedron



- **Rotational** icosahedral group is  $I = A_5$  of order **60**
- **Full** icosahedral group is  $H_3$  of order **120** (including reflections/inversion); generated by the root system icosidodecahedron

# Clifford Algebra and orthogonal transformations

- Form an algebra using the **Geometric Product** for two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- Inner product** is symmetric  $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting  $a$  in  $b$  is given by  $a' = a - 2(a \cdot b)b = -bab$  ( $b$  and  $-b$  **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal/conformal/modular transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$



# Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- We can form the elements of the Coxeter groups by **multiplying together root vectors** in this algebra  $\alpha_i \alpha_j \dots$
- General: 8 components, **even** products: (rotations/spinors)  
**four** components:

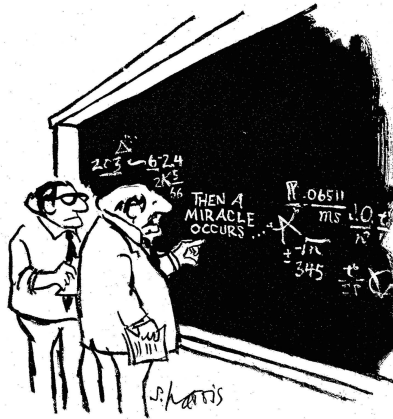
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

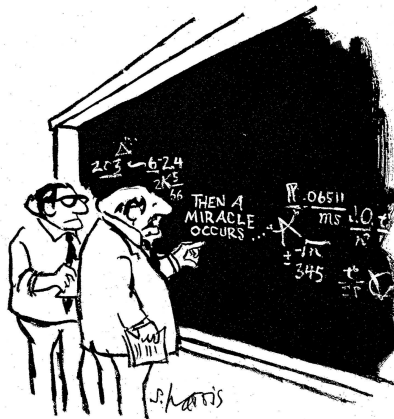
$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

- 1  $H_4$  as a rotation group I: spinor induction, Trinities and McKay correspondence
- 2 The birth of  $E_8$  out of the spinors of the icosahedron
- 3  $H_4$  as a rotation group II: The Coxeter plane

# 4D from 3D

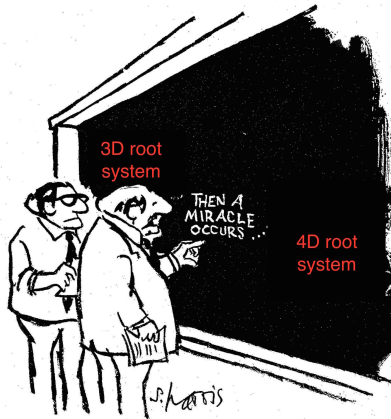


# 4D from 3D

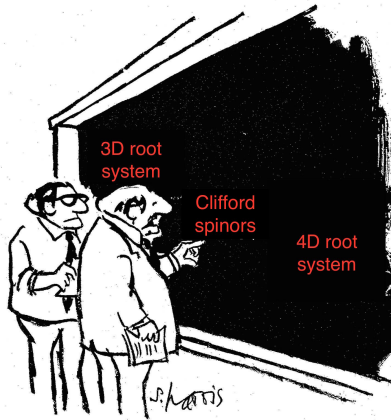


"I THINK YOU SHOULD BE  
MORE EXPLICIT HERE IN STEP TWO."

## 4D from 3D



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# Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

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- Check axioms:

1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$



# Induction Theorem – root systems

- Theorem: **3D spinor groups** give **4D root systems**.
- Check axioms:
  1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
  2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1.  **$R$  and  $-R$**  are in a spinor group by construction (**double cover** of orthogonal transformations), 2. closure under reflections is guaranteed by the **closure property of the spinor group** (with a twist:  $-R_1 \tilde{R}_2 R_1$ )
- Induction Theorem: **Every rank-3 root system induces a rank-4 root system** (and thereby **Coxeter groups**)
- Counterexample: **not every rank-4 root system** is induced in this way

# Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6 **reflections** in  $A_1 \times A_1 \times A_1$  generate 8 **spinors**.
- $\pm e_1, \pm e_2, \pm e_3$  give the 8 spinors  $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion** group  $Q$ .

# Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **reflections** in  $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$  generate 8/24/48/120 **spinors**.
- E.g.  $\pm e_1, \pm e_2, \pm e_3$  give the 8 spinors  $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion group**  $Q$  / **binary tetrahedral group**  $2T$  / **binary octahedral group**  $2O$  / **binary icosahedral group**  $2I$ ).

$A_1^3$	$A_3$	$B_3$	$H_3$
$A_1^4$	$D_4$	$F_4$	$H_4$

# Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of  $A_1 \times A_1 \times A_1 \times A_1$ ,  $D_4$ ,  $F_4$  and  $H_4$
- Exceptional phenomena:  $D_4$  (triality, important in string theory),  $F_4$  (largest lattice symmetry in 4D),  $H_4$  (largest non-crystallographic symmetry)
- Exceptional  $D_4$  and  $F_4$  arise from series  $A_3$  and  $B_3$
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

## 3D vs 4D

- Have  $A_n$ ,  $B_n$  and  $D_n$  families of root systems in **any dimension**
- In **3D**, have  **$H_3$  as an accident** (icosahedron and dodecahedron)
- In **4D**, have  $F_4$  and  $H_4$  (and in some sense  $D_4$ ) as accidents
- These 4D accidents have **unusual automorphism groups**
- Can **induce** all of these from the 3D cases, show they are **root systems** and explain their **automorphism groups**

# Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups**  $Q$ ,  $2T$ ,  $2O$  and  $2I$ , which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the '**mysterious symmetries**'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$	
$A_3$		$2T$	$D_4$	
$B_3$		$2O$	$F_4$	
$H_3$		$2I$	$H_4$	

# Induction Theorem – automorphism

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots**  $|\Phi|$  is equal to  $|G|$ , the **order of the spinor group**
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.

# Spinorial Symmetries of 4D Polytopes

## Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
$A_3$	12	24	$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$B_3$	18	48	$F_4$ lattice	48	$48^2 = 2304$
$H_3$	30	120	$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^3$	6	8	$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries**!



# Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus  $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces**  $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres**  $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles**  $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras**  $(E_6, E_7, E_8)$
- The symmetries of the **Platonic Solids**  $(A_3, B_3, H_3)$
- The **4D groups**  $(D_4, F_4, H_4)$
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

# Platonic Trinities

- Arnold's connection between  $(A_3, B_3, H_3)$  and  $(D_4, F_4, H_4)$  is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is  
 $24 = 2(1 + 3 + 3 + 5), 48 = 2(1 + 5 + 7 + 11), 120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights**  $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$  of the Coxeter groups  $(D_4, F_4, H_4)$
- Believe the Clifford connection is **more direct**

# A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar**  $I$  this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group  $H_3$  in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

## Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of pinors in Clifford algebra
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24):  $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48):  $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120):  $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- Binary groups are **discrete subgroups of  $SU(2)$**  and all thus have a  $2_s$  spinor irrep
- Connection with the **McKay correspondence!**

# The McKay Correspondence: Coxeter number, dimensions of irreps and (tensor product) graphs

binary polyhedral groups  
 $2T, 2O, 2I$   
 $\sum d_i$  12, 18, 30  
 $\sum d_i^2$  24, 48, 120

McKay correspondence

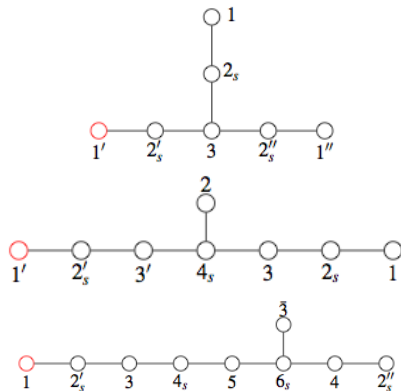
Exceptional  
Lie Groups

$E_6$ , 12

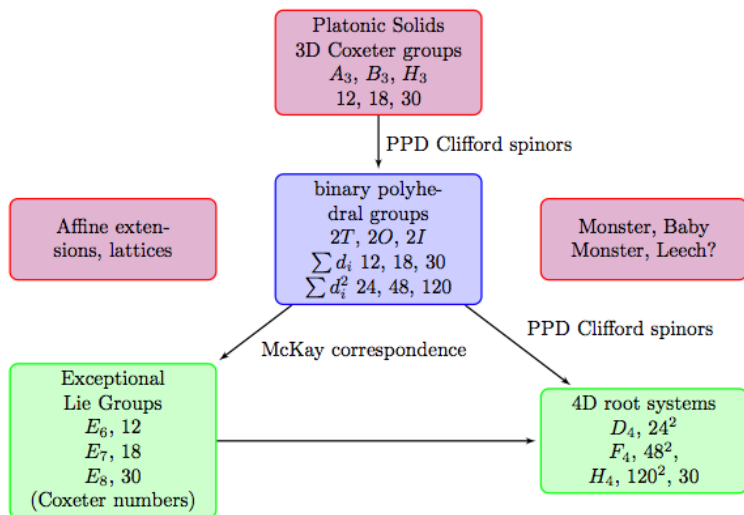
$E_7$ , 18

$E_8$ , 30

(Coxeter numbers)



# The McKay Correspondence



## 4D geometry is surprisingly important for HEP

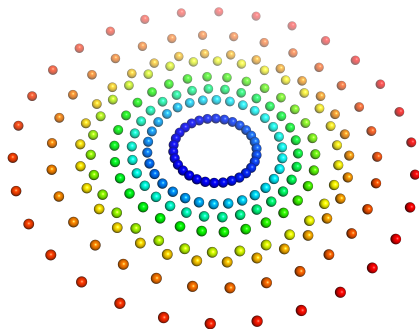
- 4D root systems are **surprisingly relevant to HEP**
- $A_4$  is  $SU(5)$  and comes up in **Grand Unification**
- $D_4$  is  $SO(8)$  and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- $B_4$  is  $SO(9)$  and is the little group of **M-Theory**
- $F_4$  is the **largest crystallographic** symmetry in 4D and  $H_4$  is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

- 1  $H_4$  as a rotation group I: spinor induction, Trinities and McKay correspondence
- 2 The birth of  $E_8$  out of the spinors of the icosahedron
- 3  $H_4$  as a rotation group II: The Coxeter plane



# Exceptional $E_8$ – the holy grail of maths and physics

- **Lie group** well-known (string theory, GUTs): triangular decomposition  $248 = 120 + 8 + 120$
- **Root system** has **240** roots – 120 creation and annihilation operators, and 8 QN/Cartan degrees of freedom





# Exceptional $E_8$ – from the icosahedron

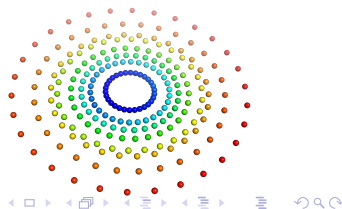
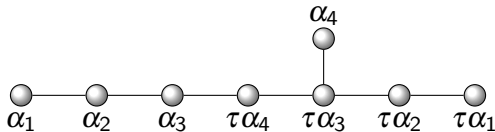
- Saw **even** products of the 30 roots of  $H_3$  gave 120 **spinors** which in turn gave  $H_4$  root system
- Taking **all** products gives group of 240 **pinors** with 8 components
- Essentially the inversion  $I$  just doubles the spinors

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

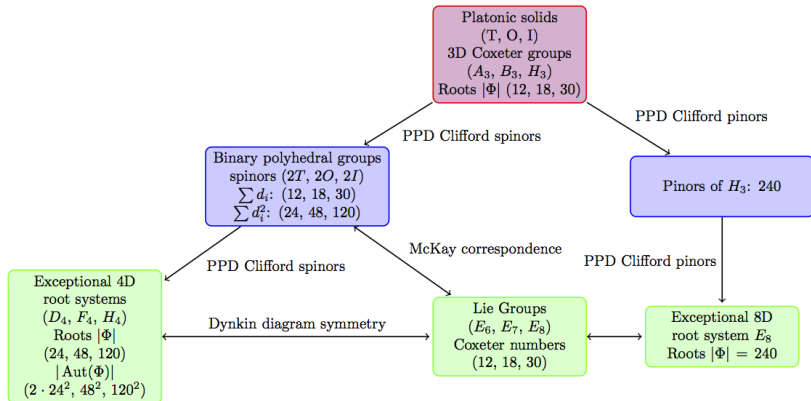
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \quad \&IR = b_0 e_1 e_2 e_3 + b_1 e_1 + b_2 e_2 + b_3 e_3$$

- Most intuitive inner product on the pinors gives only  $H_4 \oplus H_4$
- But slightly more technical **inner product** gives precisely the  **$E_8$  root system** from the **icosahedron**!

- Order 120 group  $H_3$  doubly covered by 240 (s)pinors
- Essentially  $H_4 + IH_4$ , two sets of 120
- Multiply second set by  $\tau I$ , take inner products, take into account  $\tau^2 = \tau + 1$ , but THEN: set  $\tau \rightarrow 0$ ! Each inner product is  $(\alpha_i, \alpha_j) = a + \tau b \rightarrow (\alpha_i, \alpha_j)_\tau := a$
- Like the other exceptional geometries,  $E_8$  is actually hidden within 3D geometry!

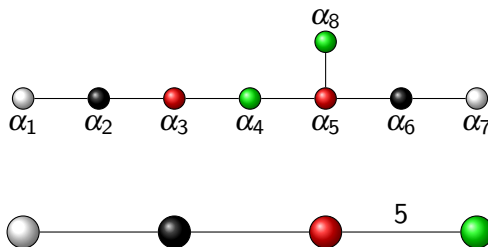


# New, explicit connections – first examples of things requiring Clifford techniques?



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# Projection and Diagram Foldings

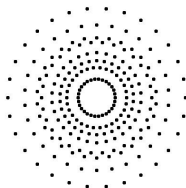
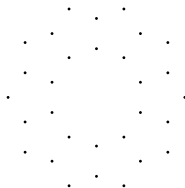
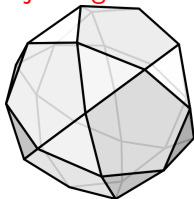


$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

- $E_8$  has a  $H_4$  subgroup of **rotations** via a '**partial folding**'
- Can **project** 240  $E_8$  roots to  $H_4 + \tau H_4$  – essentially the **reverse** of my construction!
- Coxeter element & number** of  $E_8$  and  $H_4$  are the **same**

# The Coxeter Plane

- Can show **every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane





# Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees**  $d$  are important invariants/group characteristics.
- Turns out that actually **degrees**  $d$  are intimately related to so-called **exponents**  $m$   $m = d - 1$ .

# Coxeter Elements, Degrees and Exponents

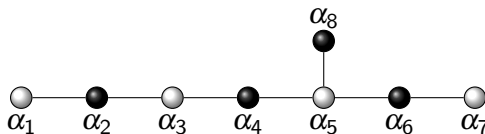
- A **Coxeter Element** is any combination of all the simple reflections  $w = s_1 \dots s_n$ , i.e. in Clifford algebra it is encoded by the versor  $W = \alpha_1 \dots \alpha_n$  acting as  $v \rightarrow wv = \pm \tilde{W} v W$ . All such elements are conjugate and thus their **order** is invariant and called the **Coxeter number**  $h$ .
- The Coxeter element has **complex eigenvalues** of the form  $\exp(2\pi mi/h)$  where  $m$  are called **exponents**:  
 $w x = \exp(2\pi mi/h) x$
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real sections** again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

# Coxeter Elements, Degrees and Exponents

- The Coxeter element has **complex eigenvalues** of the form  $\exp(2\pi mi/h)$  where  $m$  are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, **1** and  **$h-1$**  are always exponents
- Turns out that actually **exponents and degrees** are intimately related ( $m = d - 1$ ). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

# The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (**eigenblades**) rather than just eigenvectors
- In particular, can show **every** (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an **alternate colouring**
- Essentially just gives **two sets of mutually commuting generators**



# The Coxeter Plane

- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an alternate colouring
- Essentially just gives **two sets of orthogonal = mutually commuting generators but anticommuting root vectors**  $\alpha_w$  and  $\alpha_b$  (duals  $\omega$ )
- Cartan matrices are positive definite, and thus have a **Perron-Frobenius** (all positive) eigenvector  $\lambda_i$ .
- Take **linear combinations** of components of this eigenvector as coefficients of two vectors from the orthogonal sets  

$$v_w = \sum \lambda_w \omega_w \text{ and } v_b = \sum \lambda_b \omega_b$$
- Their **outer product/Coxeter plane bivector**  $B_C = v_b \wedge v_w$  describes an **invariant plane** where  $w$  acts by rotation by  $2\pi/h$ .

# Clifford Algebra and the Coxeter Plane – 2D case

$$I_2(n) \quad \circ \xrightarrow{n} \circ$$

- For  $I_2(n)$  take  $\alpha_1 = e_1, \alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$

- So **Coxeter versor** is just

$$W = \alpha_1 \alpha_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} e_1 e_2 = -\exp \left( -\frac{\pi I}{n} \right)$$

- In Clifford algebra it is therefore immediately obvious that the action of the  $I_2(n)$  Coxeter element is described by a versor (here a rotor/spinor) that encodes **rotations in the  $e_1 e_2$ -Coxeter-plane** and yields  $h = n$  since trivially  $W^n = (-1)^{n+1}$  yielding  $w^n = 1$  via  $wv = \tilde{W}vW$ .

## Clifford Algebra and the Coxeter Plane – 2D case

- So **Coxeter versor** is just

$$W = -\exp\left(-\frac{\pi I}{n}\right)$$

- $I = e_1 e_2$  **anticommutes** with both  $e_1$  and  $e_2$  such that **sandwiching formula** becomes

$$v \rightarrow wv = \tilde{W}vW = \tilde{W}^2v = \exp\left(\pm\frac{2\pi I}{n}\right)v \text{ immediately}$$

yielding the standard result for the **complex eigenvalues** in real Clifford algebra **without any need for artificial complexification**

- The Coxeter plane bivector  $B_C = e_1 e_2 = I$  gives the **complex structure**
- The Coxeter plane bivector  $B_C$  is invariant under the **Coxeter versor**  $\tilde{W}B_CW = \pm B_C$ .

# Clifford Algebra and the Coxeter Plane – 3D case

- In 3D,  $A_3$ ,  $B_3$ ,  $H_3$  have  $\{1, 2, 3\}$ ,  $\{1, 3, 5\}$  and  $\{1, 5, 9\}$
- Coxeter element is product of a **spinor** in the Coxeter plane with the same complex structure as before, and a **reflection perpendicular** to the plane
- So in 3D still completely determined by the plane
- **1** and  **$h-1$**  are **rotations** in **Coxeter plane**
- **$h/2$**  is the **reflection** (for  $v$  in the normal direction)

$$wv = \tilde{W}^2 = \exp\left(\pm \frac{2\pi i}{h} \frac{h}{2}\right) = \exp(\pm \pi i)v = -v$$



# Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise**  $W$  into **orthogonal** (commuting/anticommuting) components

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n \text{ with } W_i = \exp(\pi m_i l_i / h)$$

- Here,  $l_i$  is a bivector describing a **plane** with  $l_i^2 = -1$

- For  $v$  **orthogonal to the plane** described by  $l_i$  we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v \text{ so cancels out}$$

- For  $v$  **in the plane** we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$

- Thus if we **decompose**  $W$  into **orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

# Clifford algebra: no need for complexification

- For  $v$  in the plane we have

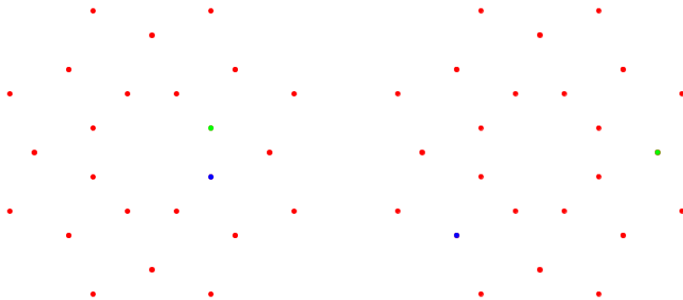
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$

- So **complex eigenvalue equation** arises geometrically **without any need** for complexification
- Different complex structures** immediately give different **eigenplanes**
- Eigenvalues/angles/**exponents** given from just factorising  $W = \alpha_1 \dots \alpha_n$
- E.g.  $B_4$  has exponents 1, 3, 5, 7 and  $W = \exp(\frac{\pi}{8} l_1) \exp(\frac{3\pi}{8} l_2)$
- Here we have been looking for orthogonal eigenspaces, so **innocuous** – different complex structures commute
- But not in general – **naive complexification** can be misleading

## 4D case: $B_4$

- E.g.  $B_4$  has exponents 1, 3, 5, 7
- Coxeter versor decomposes into **orthogonal components**

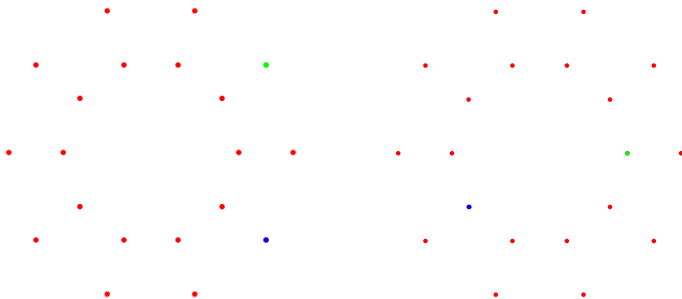
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$$



## 4D case: $A_4$

- E.g.  $A_4$  has exponents 1, 2, 3, 4
- Coxeter versor decomposes into **orthogonal components**

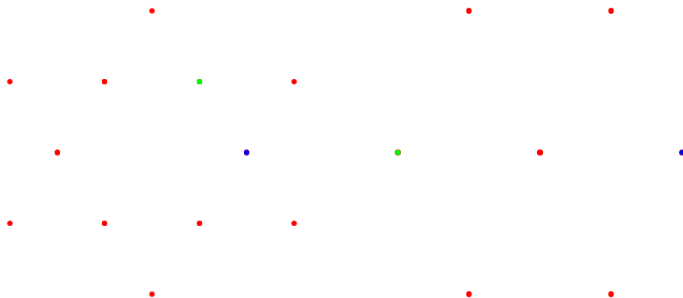
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$$



## 4D case: $D_4$

- E.g.  $D_4$  has exponents 1, 3, 3, 5
- Coxeter versor decomposes into **orthogonal components**

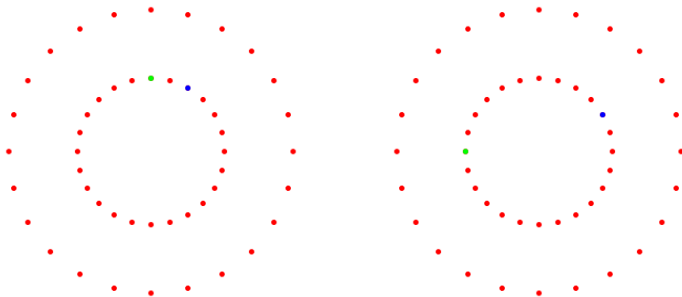
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{3\pi}{6} I B_C\right)$$



## 4D case: $F_4$

- E.g.  $F_4$  has exponents 1, 5, 7, 11
- Coxeter versor decomposes into **orthogonal components**

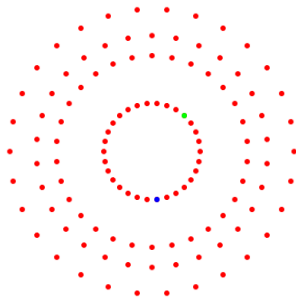
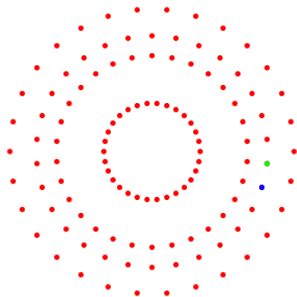
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$$



## 4D case: $H_4$

- E.g.  $H_4$  has exponents 1, 11, 19, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$$



# Clifford Algebra and the Coxeter Plane – 4D case summary

rank 4	exponents	W-factorisation
$A_4$	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
$B_4$	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
$D_4$	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
$F_4$	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
$H_4$	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way



# Clifford Algebra and the Coxeter Plane – $D_6$

- For  $D_6$  one has exponents  $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

$$W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)$$

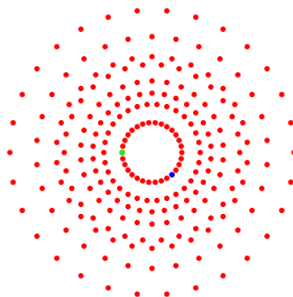
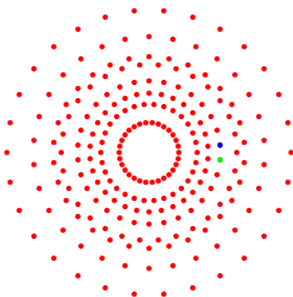
- Now **bivector exponentials** correspond to **rotations in orthogonal planes**
- **Vector** factors correspond to **reflections**
- For odd  $n$ , there is always **one such vector factor** in  $D_n$ , and for even  $n$  there are **two**

## 8D case: $E_8$

- E.g.  $H_4$  has exponents 1, 11, 19, 29,  $E_8$  has 1, 7, 11, 13, 17, 19, 23, 29

- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$

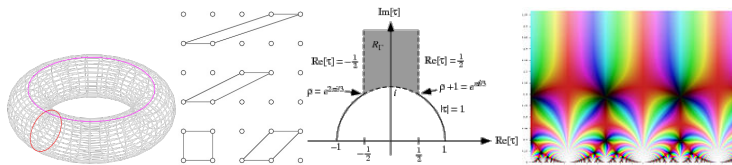


## Imaginary differences – different imaginaries

So what has been **gained** by this **Clifford view**?

- There are **different** entities that serve as **unit imaginaries**
- They have a **geometric** interpretation as an **eigenplane of the Coxeter element**
- These don't need to **commute** with everything like  $i$  (though they do here – at least anticommute. But that is because we looked for **orthogonal decompositions**)
- But see that in general **naive complexification** can be a dangerous thing to do – **unnecessary**, issues of **commutativity**, **confusing** different imaginaries etc

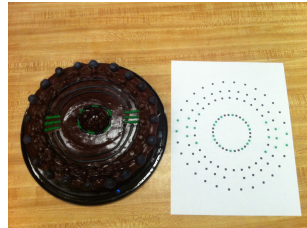
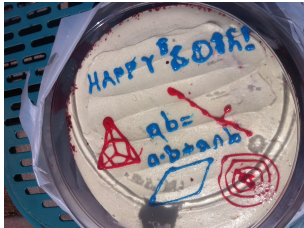
# Modular group



- **Modular group**: interested in modular forms for applications in **Moonshine/string theory**: Monster 196883, Klein  $j$  196884
- Modular generators:  $T : \tau \rightarrow \tau + 1$ ,  $S : \tau \rightarrow -1/\tau$
- $\langle S, T | S^2 = I, (ST)^3 = I \rangle$
- CGA:  $T_X = 1 + \frac{ne_1}{2}$  and  $S_X = e_1 e$
- $(S_X T_X)^3 = -1$  and  $S_X^2 = 1$

# Conclusions

- **All exceptional** geometries arise in **3D**, root systems giving rise to Lie groups/algebras etc
- Completely novel **spinorial** way of viewing the geometries as 3D phenomena – implications for HEP etc?
- More **natural** point of view, explaining **existence** and **automorphism groups**
- Totally unclear how one would see this in a **matrix framework** – might **require** Clifford point of view
- New view of Coxeter **degrees and exponents** with **geometric interpretation of imaginaries**
- A unified framework for doing **group and representation theory**: polyhedral, orthogonal, conformal, modular (Moonshine) etc



Thank you!  
 Congratulations David and Nancy!